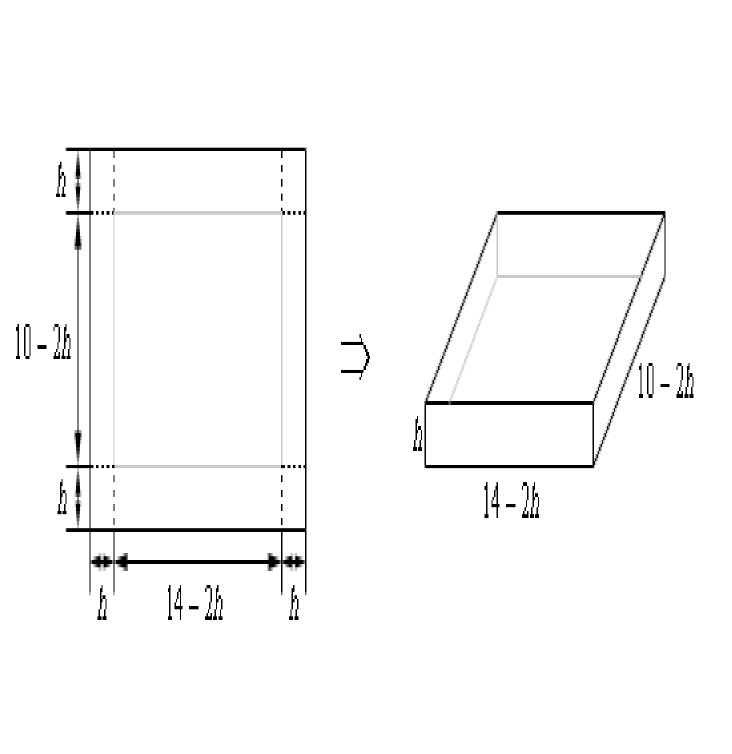
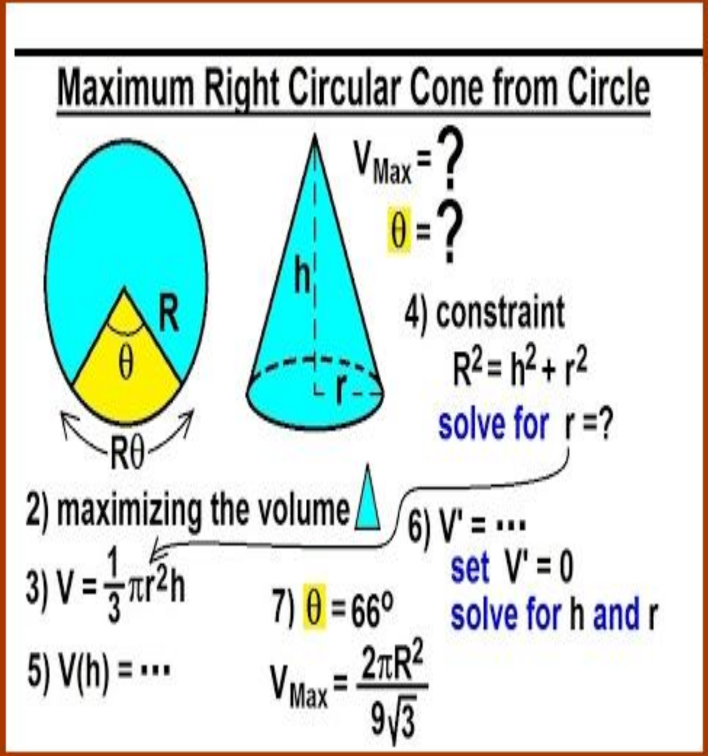
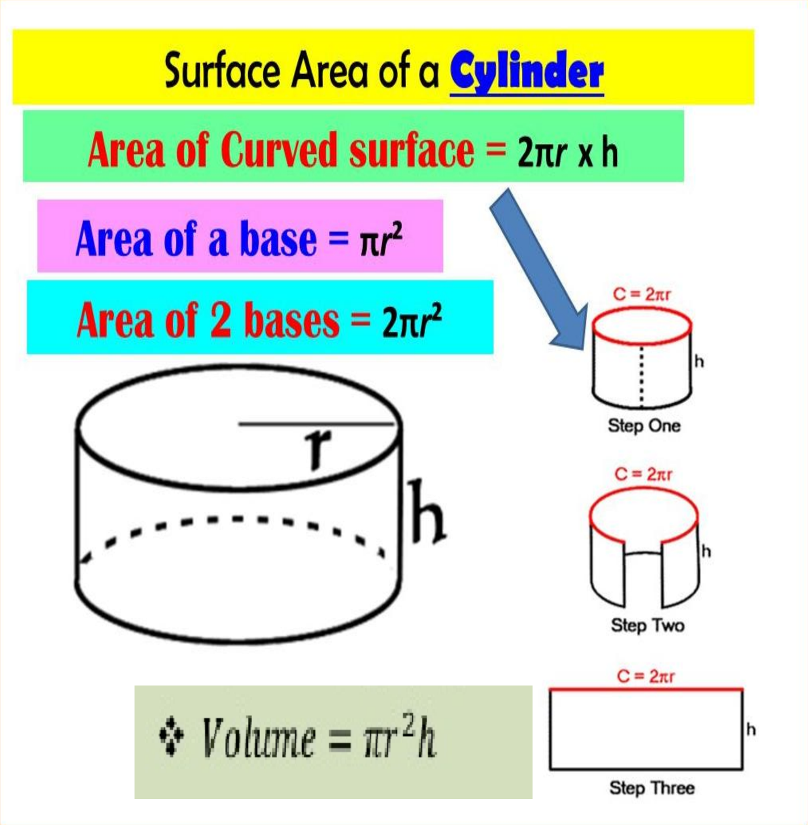
APPLICATION OF DERIVATIVES-MAX AND MIN MODULE-10







Example 5 :

A wire 12 m long is cut into two pieces so as to make one square and one circle. In order to cut in such a way so that we get the maximum area, the area function expressed in terms of **side of the square** is

$$A(s) = \pi\left(\frac{6-2s}{\pi}\right)^2 + s^2$$

How?

Solution : Wire,12 cm long is cut into

Let s be the side of the square and R be the radius of the circle.

Now total length of wire is equal to the perimeter of square + circumference of the circle

Using perimeter

$L = 4s + 2\pi R$

$12 = 4s + 2\pi R$

$R = \frac{6-2s}{\pi}$

Now total Area

$A = A_1 + A_2$

$A = \pi R^2 + s^2$

$A = \pi\left(\frac{6-2s}{\pi}\right)^2 + s^2$

Question

A Wire of length 28 m is to be cut into two pieces. One of the pieces is to be made into a square and the other into a circle. What should be the length of the two pieces so that the combined area of the circle is minimum?

Solution

Let a piece of length l be cut from the given wire to make a square.

Then, the other piece of wire to be made into a circle is of length $(28-l)m$.

Now, side of square= $\frac{l}{4}$

Let r be the radius of the circle. Then, $2\pi r = 28 - l \Rightarrow r = \frac{1}{2\pi}(28-l)$.

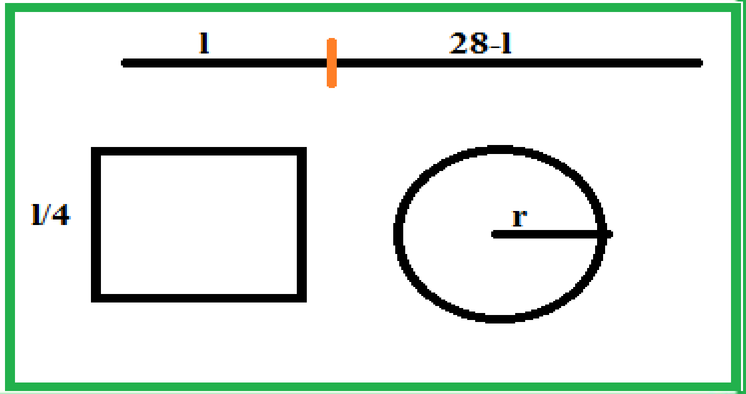
The combined areas of the square and the circle (A) is given by,

A = (side of the square)^2 + πr^2

$$= \frac{l^2}{16} + \pi\left[\frac{1}{2\pi}(28-l)\right]^2$$

$$= \frac{l^2}{16} + \frac{1}{4\pi}(28-l)^2$$

$$\therefore \frac{dA}{dl} = \frac{2l}{16} + \frac{2}{4\pi}(28-l)(-1) = \frac{l}{8} - \frac{1}{2\pi}(28-l)$$



Now, $\frac{dA}{dl} = 0 \Rightarrow \frac{l}{8} - \frac{1}{2\pi}(28 - l) = 0$

$$\Rightarrow \frac{\pi l - 4(28 - l)}{8\pi} = 0$$

$$(\pi + 4)l - 112 = 0$$

$$\Rightarrow l = \frac{112}{\pi + 4} \qquad \frac{d^2A}{dl^2} = \frac{l}{8} + \frac{1}{2\pi} > 0$$

Thus, when $l = \frac{112}{\pi + 4}$, $\frac{d^2A}{dl^2} > 0$

$\therefore$ By second derivative test, the area (A) is the minimum when $l = \frac{112}{\pi + 4}$.

Hence, the combined area is the minimum when the length of the wire in making the square is $\frac{112}{\pi + 4}$ cm while the length of the wire in making the circle is $28 - \frac{112}{\pi + 4} = \frac{28\pi}{\pi + 4}$ cm.

► **Example 4** A closed cylindrical can is to hold 1 liter ($1000\ cm^3$) of liquid. In order to find the height and radius to minimize the amount of material needed to manufacture the can, Show that total area can be modelled as

$$A(r) = 2\pi r^2 + \frac{2000}{r}$$

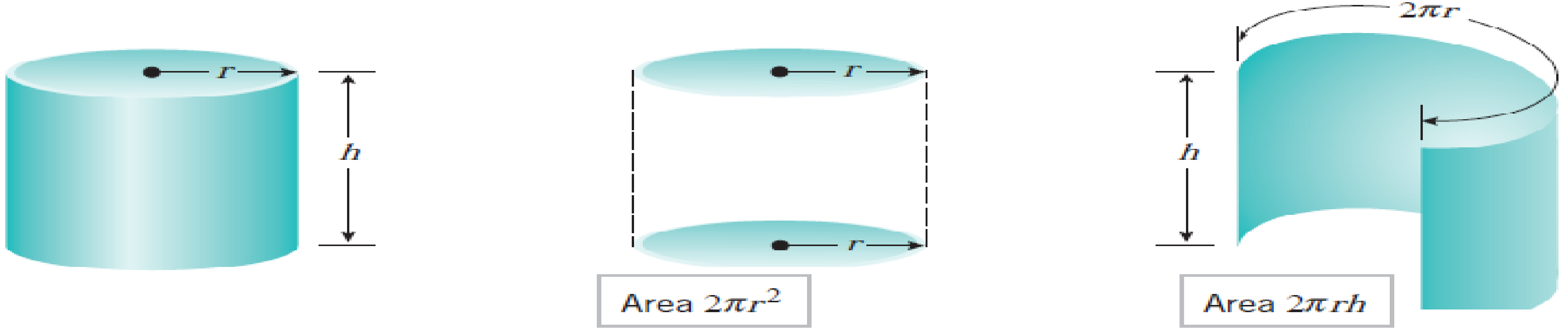
Solution.

Let h be height (in cm), r be radius (in cm) & S be surface area (in cm^2) of the can

Assuming there is no waste or overlap, the amount of material needed for manufacture will be the same as the surface area of the can. Since the can consists of two circular disks of radius r and a rectangular sheet with dimensions h by $2\pi r$ (Figure 4.5.7), the surface area will be

$$S = 2\pi r^2 + 2\pi r h \qquad (1)$$

Since S depends on two variables, r and h, we will look for some condition in the problem that will allow us to express one of these variables in terms of the other. For this purpose,



Now for the volume of a cylinder $\quad 1000 = \pi r^2 h \quad$ or $\quad h = \dfrac{1000}{\pi r^2} \qquad (2)$

Using (1) and (2) $\quad S = 2\pi r^2 + \dfrac{2000}{r} \qquad (3)$

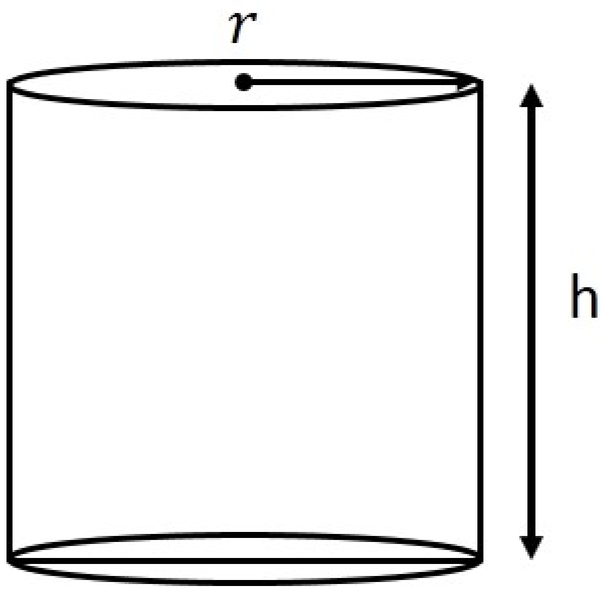
Question

Show that the right circular cylinder of given surface and maximum volume is such that is heights is equal to the diameter of the base.

Solution

Let r and h be the radius and height of the cylinder respectively.

Then, the surface area (S) of the cylinder is given by,



$$S = 2\pi r^2 + 2\pi rh$$

$$\Rightarrow h = \frac{S - 2\pi r^2}{2\pi r}$$

$$= \frac{S}{2\pi}\left(\frac{1}{r}\right) - r$$

Let V be the volume of the cylinder. Then,

$$V = \pi r^2 h = \pi r^2\left[\frac{S}{2\pi}\left(\frac{1}{r}\right) - r\right] = \frac{Sr}{2} - \pi r^3$$

Let V be the volume of the cylinder. Then,

$$V = \pi r^2 h = \pi r^2\left[\frac{S}{2\pi}\left(\frac{1}{r}\right) - r\right] = \frac{Sr}{2} - \pi r^3$$

Then, $\frac{dV}{dr} = \frac{S}{2} - 3\pi r^2$,

Now, **For max V,** $\frac{dV}{dr} = 0 \Rightarrow \frac{S}{2} = 3\pi r^2 \Rightarrow r^2 = \frac{S}{6\pi}$

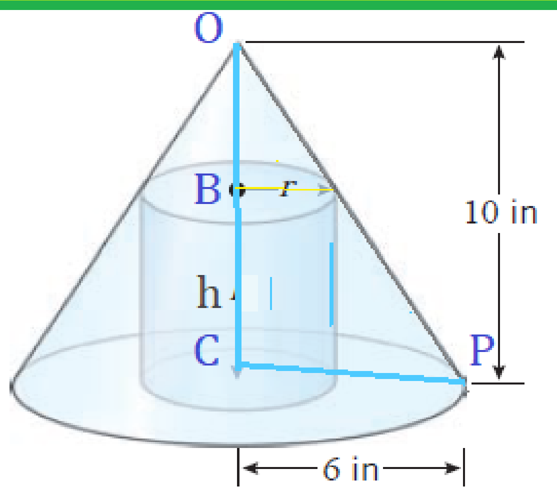
$$\frac{d^2V}{dr^2} = -6\pi r \; < 0$$

When $r^2 = \frac{S}{6\pi}$, then $\frac{d^2V}{dr^2} = -6\pi\left(\sqrt{\frac{S}{6\pi}}\right) < 0$.

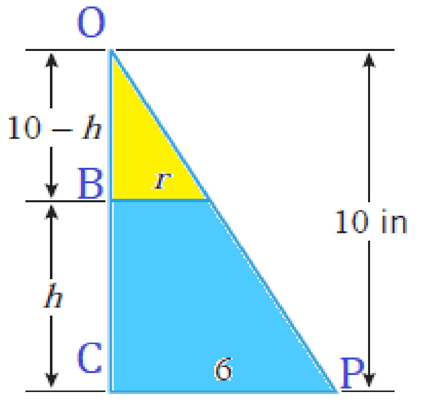
By Second derivative test, the volume is the maximum when $r^2 = \frac{S}{6\pi}$.

Now, when $r^2 = \frac{S}{6\pi}$, then $h = \frac{6\pi r^2}{2\pi}\left(\frac{1}{r}\right) - r = 3r - r = 2r$

Hence, the volume is the maximum when the height is twice the radius i.e., when the height is equal to the diameter.



(a)



(b)

► **Example 3** In order to find the radius and height of the right circular cylinder of largest volume that can be inscribed in a right circular cone with radius 6 inches and height 10 inches the volume function in terms of radius is

$$V(r) = 10\pi r^2 - \frac{5}{3}\pi r^3$$

Solution.

Let

r = radius (in inches) of the cylinder

h = height (in inches) of the cylinder

V = volume (in cubic inches) of the cylinder

The formula for the volume of the inscribed cylinder is $V = \pi r^2 h$ (1)

To eliminate one of the variables in (1) we need a relationship between r and h. Using similar triangles (Figure b) we obtain

$$\frac{10-h}{r} = \frac{10}{6} \quad \text{or} \quad h = 10 - \frac{5}{3}r \qquad (2)$$

Substituting (2) into (1) we obtain

$$V = \pi r^2\left(10 - \frac{5}{3}r\right) = 10\pi r^2 - \frac{5}{3}\pi r^3 \qquad (3)$$

which expresses V in terms of r alone. Because r represents a radius, it cannot be negative, and because the radius of the inscribed cylinder cannot exceed the radius of the cone, the variable r must satisfy

$$0 \le r \le 6$$

Prove that the volume of the largest cone that can be inscribed in a

Cone of largest volume inscribed in the sphere of radius R

Let OC = x

Radius of cone = BC $= \mathbf{r}$

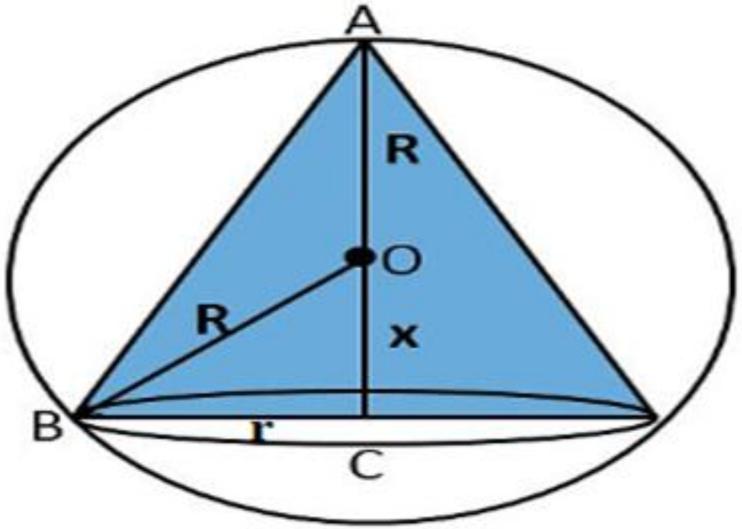
Height of cone = h = OC + OA

$= \mathbf{R + x}$

Δ BOC is a right angled triangle

$$\mathbf{r^2 = R^2 - x^2}$$

$$\mathbf{v} = \frac{1}{3}\pi\, \mathbf{r^2\, h}$$



$$V = \frac{1}{3}\pi(R^2 - x^2)(R + x) = \frac{1}{3}\pi(R+x)^2(R-x)$$

$$\frac{dV}{dx} = \frac{\pi}{3}\,(R+x)^2(-1) + 2(R+x)(R-x) = \frac{1}{3}\pi(R+x)[(R+x)(-1)+2(R-x)]$$

$$= \frac{1}{3}\pi\,(R + x)(R - 3x)$$

For max V, $\frac{dV}{dx} = 0$

, $x = -R$ & $x = \frac{R}{3}$

Since x cannot be negative

$$x = \frac{R}{3}$$

$$\frac{d^2v}{dx^2} = \frac{1}{3}\pi\,(-3(R+x)+(R-3x))$$

$$\frac{d^2v}{dx^2} = \frac{\pi}{3}(-2R - 6x)$$

$$\frac{d^2v}{dx^2} = \frac{-\pi}{3}(2R + 6x)$$

Putting $x = \frac{R}{3}$

Thus $\frac{d^2v}{dx^2} < 0$ when $x = \frac{R}{3}$

∴ Volume is Maximum when $x = \frac{R}{3}$

Volume of cone = $\frac{1}{3}\pi(R^2 - x^2)(R + x)$

Putting $x = \frac{R}{3}$

$$= \frac{1}{3}\pi\left(R^2 - \left(\frac{R}{3}\right)^2\right)\left(R + \frac{R}{3}\right)$$

$$= \frac{1}{3}\pi\left(R^2 - \frac{R^2}{9}\right)\left(\frac{3R + R}{3}\right)$$

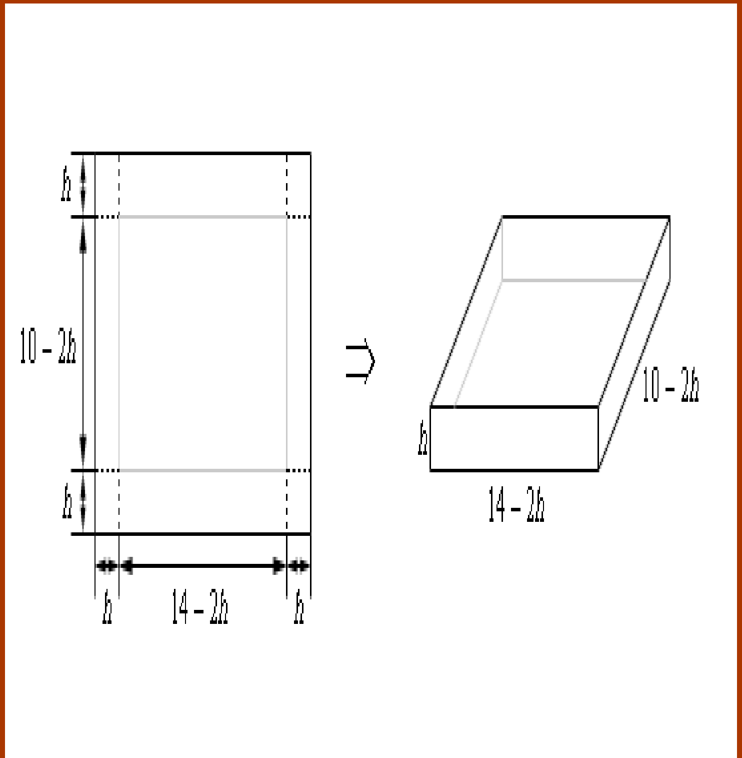
$$= \frac{1}{3}\pi\left(\frac{9R^2 - R^2}{9}\right)\left(\frac{4R}{3}\right)$$

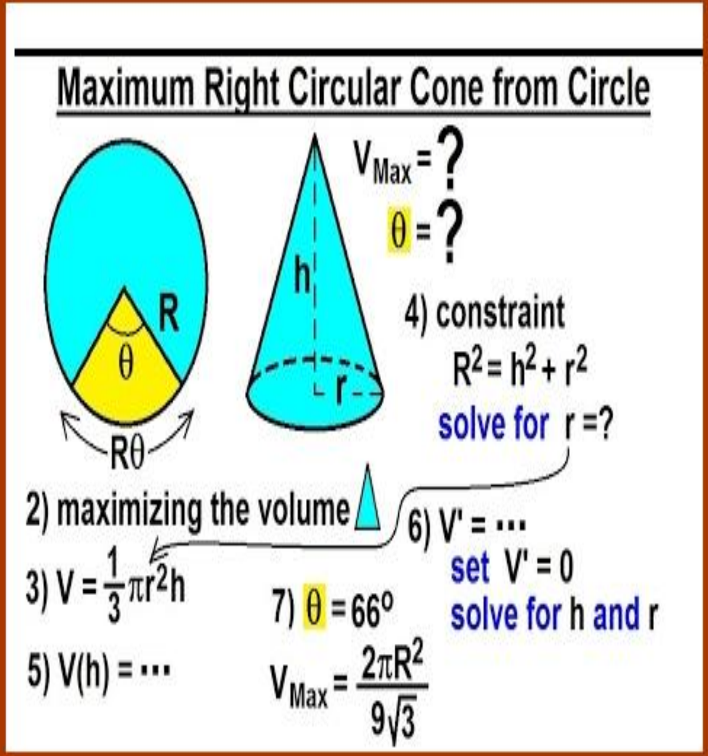
$$= \frac{32}{81}\pi R^3 = \frac{8}{27}\left(\frac{4}{3}\pi r^3\right)$$

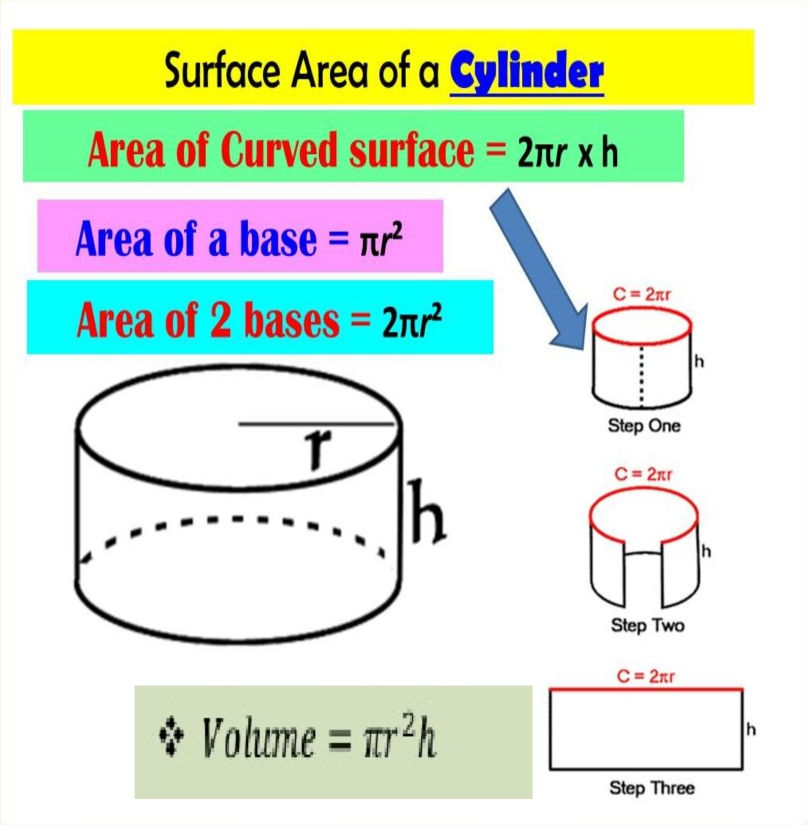
HOME ASSIGNMENT

- EX 6.5
- Q 5 (I), (II),(III), (IV)
- Q6 , Q 7 , Q 8

APPLICATION OF DERIVATIVES-MODULE 11



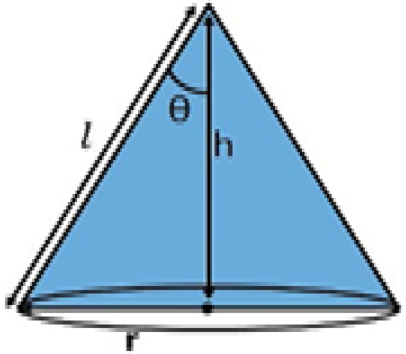




Ex 6.5, 25

Show that the semi-vertical angle of the cone of the maximum volume and of given slant height is $\tan^{-1}\sqrt{2}$

Let l be the slant height & θ be the semi vertical angle of the cone.



We need to maximize volume of cone

$$V = \frac{1}{3}\pi r^2 h$$

$$= \frac{\pi}{3}(l^2 - h^2)h$$

$$r^2 = l^2 - h^2$$

$$= \frac{\pi}{3}(l^2 h - h^3)$$

$$\frac{dV}{dh} = \frac{\pi}{3}(l^2 - 3h^2) \qquad \text{For max.V, } \frac{dV}{dh} = 0$$

$$l^2 = 3h^2$$

$$h = \frac{l}{\sqrt{3}} \quad \text{------(1)}$$

$$r^2 = l^2 - h^2$$

$$r = \frac{\sqrt{2}\, l}{\sqrt{3}} \quad \text{---------(2)}$$

$$\tan\theta = r/h = \frac{\frac{\sqrt{2}}{\sqrt{3}}\, l}{\frac{1}{\sqrt{3}}\, l}$$

$$\tan\theta = \sqrt{2}$$

$$\frac{d^2y}{dx^2} = -\ \pi h^2$$

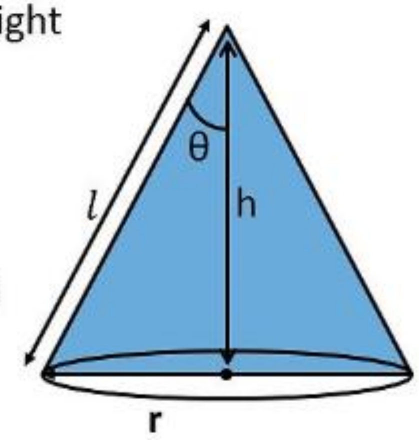
$$< 0$$

Volume is maximum ,when $\tan\theta = \sqrt{2}$

Ex 6.5, 26

Show that semi-vertical angle of right circular cone of given surface area and maximum volume is $\sin^{-1}\left(\frac{1}{3}\right)$

Let r , h & l be the radius, height & slant height of a cone respectively



And Let V & S be the volume & surface area & θ be a semi vertical angle of a cone

Given surface Area of a cone is constant

Surface Area of a cone $= \pi r^2 + \pi r l$

$$\frac{S - \pi r^2}{\pi r} = l \qquad ...(1)$$

$$l = \frac{S - \pi r^2}{\pi r}$$

Volume of a cone $= \frac{1}{3}\pi r^2 h$

h= $\sqrt{l^2 - r^2}$

$$V = \frac{1}{3}\pi r^2 \sqrt{l^2 - r^2}$$

$$V = \frac{1}{3}\pi r^2 \sqrt{\left(\frac{S - \pi r^2}{\pi r}\right)^2 - r^2} \qquad \text{(From (1))}$$

$$V = \frac{1}{3}\pi r^2 \sqrt{\frac{(S - \pi r^2)^2 - \pi^2 r^2 (r^2)}{\pi^2 r^2}}$$

$$V = \frac{r}{3}\sqrt{S^2 + \pi^2 r^4 - 2S\pi r^2 - \pi^2 r^4}$$

$$V = \frac{r}{3}\sqrt{S^2 - 2\,S\pi r^2}$$

Since V has square root

It will be difficult to differentiate

So, we take

$$Z = V^2$$

$$Z = \frac{1}{9}(r^2 S^2 - 2\,S\pi r^4)$$

Diff. Z w.r.t r

$$\frac{dZ}{dr} = \frac{1}{9}[s^2(2r) - 2s\pi\,(4r^3)]$$

$$\frac{dZ}{dr} = \frac{1}{9}[2rs^2 - 8s\pi r^3]$$

For maximum volume, $\frac{dZ}{dr} = 0$

$$\frac{1}{3}[2rs^2 - 8s\pi r^3] = 0$$

$$2rs^2 - 8s\pi r^3 = 0$$

$$2rs^2 = 8s\pi r^3$$

$$s = 4\pi r^2 \quad \text{———} \quad (2)$$

Finding $\frac{d^2Z}{dr^2}$

$$\frac{dZ}{dr} = \frac{1}{9}[2rs^2 - 8s\pi r^3]$$

$$\frac{d^2Z}{dr^2} = \frac{1}{9}[2s^2 - 8s\pi(3r^2)]$$

$$\frac{d^2Z}{dr^2} = \frac{1}{9}[2s^2 - 24s\pi r^2]$$

Putting $s = 4\pi r^2$

$$\frac{d^2Z}{dr^2} = \frac{1}{9}[-64\pi^2 r^4] < 0$$

Volume is maximum for $s = 4\pi r^2$

Compare (1) and (2)

$$4\pi r^2 = \pi r^2 + \pi r l$$

$$3\pi r^2 = \pi r l$$

$$l = 3r$$

$$\frac{l}{r} = 3$$

$$\sin\theta = \frac{r}{l} = \frac{1}{3}$$

$$\theta = \sin^{-1}\frac{1}{3}$$

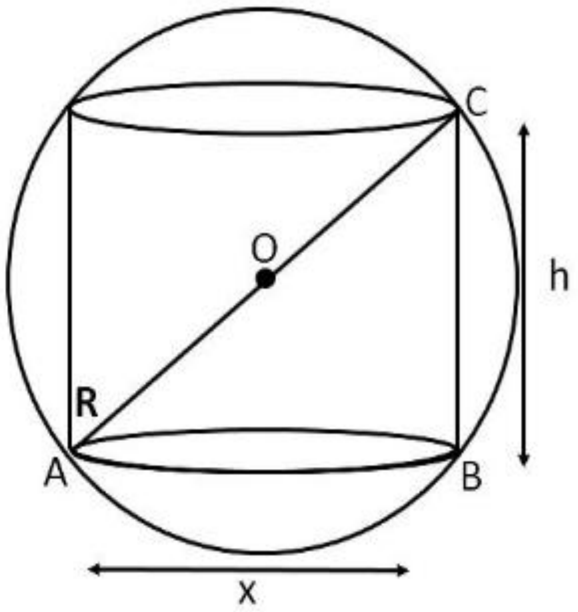
Misc 17

Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$. Also find the maximum volume.

Let R be the radius of sphere

Let h be the height

& x be the diameter of cylinder



In ΔABC

Using Pythagoras theorem

$$(CB)^2 + (AB)^2 = (AC)^2$$

$$h^2 + (x)^2 = (R + R)^2$$

$$h^2 + x^2 = (2R)^2$$

$$h^2 + x^2 = 4R^2$$

$$x^2 = 4R^2 - h^2 \quad ...(1)$$

We need to find maximum volume of cylinder

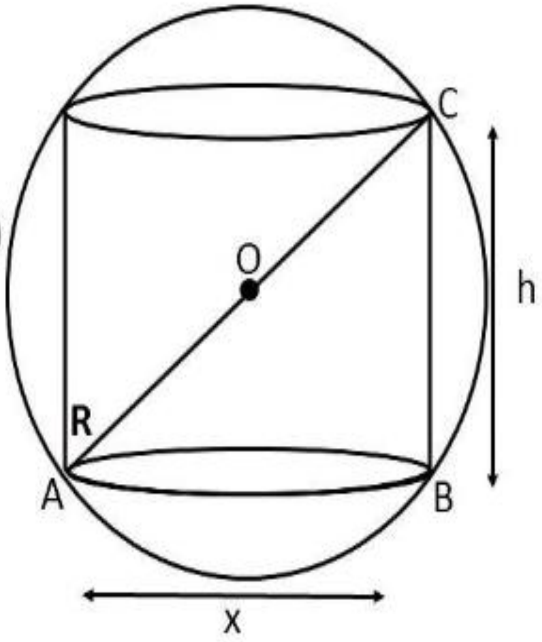
Let V be the volume of cylinder

$$V = \pi\, (radius)^2 \times (height)$$

$$V = \pi \left(\frac{x}{2}\right)^2 \times h$$

$$V = \pi \times \frac{x^2}{4} \times h$$

$$V = \pi \frac{(4R^2 - h^2)}{4} \times h \quad \text{(From (1))}$$



$$V = \pi \frac{(4R^2 - h^2)}{4} \times h$$

$$V = \frac{4R^2\pi h}{4} - \frac{\pi h^3}{4}$$

$$V = \pi h R^2 - \frac{\pi h^3}{4}$$

$$\frac{dv}{dh} = \pi R^2 - \frac{\pi}{4}(3h^2)$$

$$\frac{dv}{dh} = \pi R^2 - \frac{3\pi}{4} h^2$$

Putting $\frac{dv}{dh} = 0$

$$\pi R^2 - \frac{3}{4}\pi h^2 = 0$$

$$h^2 = \frac{4R^2}{3}$$

$$h = \frac{2R}{\sqrt{3}}$$

So, volume is maximum when $h = \frac{2R}{\sqrt{3}}$

Finding $\frac{d^2v}{dh^2} = \frac{-3\pi h}{2}$ *< 0*

VOLUME is maximum

$$x^2 = 4R^2 - h^2$$

$$x^2 = 4R^2 - \left(\frac{2R}{\sqrt{3}}\right)^2$$

$$x^2 = 4R^2 - \frac{4R^2}{3}$$

$$x^2 = \frac{8}{3}R^2$$

Putting value of x^2 & h

$$V = \frac{\pi}{4} \times \frac{8R^2}{3} \times \frac{2R}{\sqrt{3}}$$

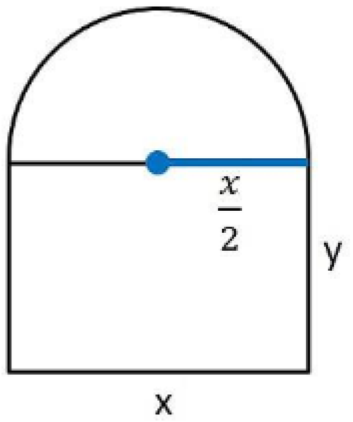
$$V = \frac{16\pi R^3}{12\sqrt{3}}$$

Misc 11, Q 11

A window is in the form of a rectangle surmounted by a semicircular opening. The total perimeter of the window is 10 m. Find the dimensions of the window to admit maximum light through the whole opening.

Let Length of rectangle be x

& breadth of rectangle be y



Diameter of semicircle = x

∴ Radius of semicircle = $\frac{x}{2}$

Given ,

Perimeter of window = 10 m

Length + 2 × Breadth + circumference of semicircle = 10

$$x + 2y + \pi\left(\frac{x}{2}\right) = 10$$

$$2y = 10 - x - \frac{\pi x}{2}$$

$$y = \frac{10}{2} - \frac{x}{2} - \frac{1}{2} \times \frac{\pi x}{2}$$

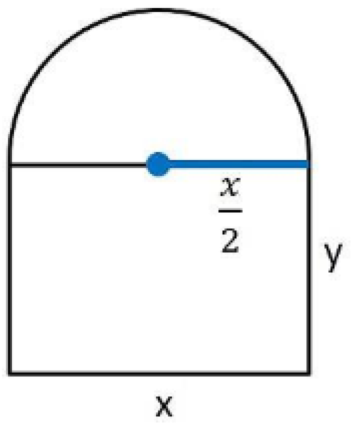
$$y = 5 - x\left(\frac{1}{2} + \frac{\pi}{4}\right) \qquad ...(1)$$

We need to maximize area of window

Area of window = Area of rectangle + Area of Semicircle

$$A = \text{Length} \times \text{Breadth} + \frac{1}{2} \times \pi r^2$$

$$A = xy + \frac{1}{2} \times \pi \left(\frac{x}{2}\right)^2$$



Putting value of y from (1)

$$A = x\left(5 - x\left(\frac{1}{2} + \frac{\pi}{4}\right)\right) + \frac{1}{2} \times \frac{\pi x^2}{4}$$

$$A = 5x - \frac{1}{2}x^2 - \frac{\pi x^2}{4} + \frac{\pi x^2}{8}$$

$$A = 5x - \frac{1}{2}x^2 - \frac{\pi x^2}{8}$$

$$\frac{dA}{dx} = 5 - x - \frac{\pi x}{4}$$

Putting $\frac{dA}{dx} = 0$

$$0 = 5 - x - \frac{\pi x}{4}$$

$$x + \frac{\pi x}{4} = 5$$

$$x = \frac{5}{\left(1 + \frac{\pi}{4}\right)}$$

$$x = \frac{20}{\pi + 4}$$

$$\frac{d^2A}{dx^2} = -1 - \frac{\pi}{4}$$

$$< 0$$

Hence, A is maximum when $x = \frac{20}{\pi + 4}$

$$y = 5 - \frac{20}{\pi + 4}\left(\frac{2 + \pi}{4}\right)$$

$$y = \frac{10}{\pi + 4}$$

Misc 18, Q 18

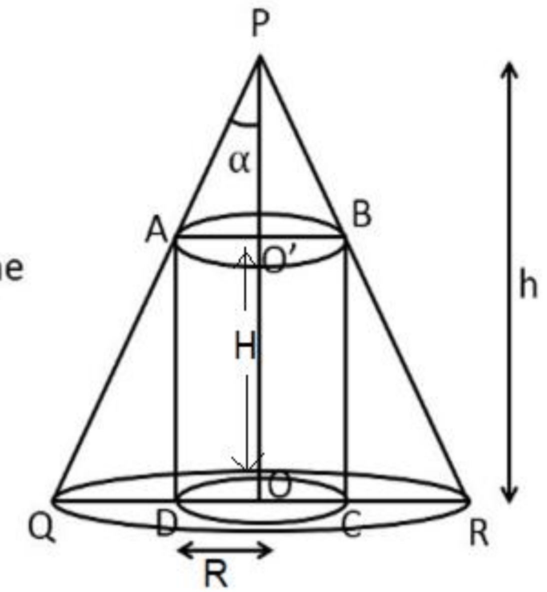
Show that height of the cylinder of greatest volume which can be inscribed in a right circular cone of height h and semi vertical angle α is one-third that of the cone and the greatest volume of cylinder is $\frac{4}{27}\pi h^3 \tan^2 \alpha$

Let PQR be the cone of height h

i.e. PO = h

& α be the semi vertical angle of cone

Let R be the radius of cylinder ABCD and H be the height.



Let V be the volume of cylinder

V = π $(radius)^2 (height)$

$$\tan\alpha = \frac{R}{h-H}$$

$$R = \tan\alpha(h-H)$$

$$V = \pi R^2 H$$
$$= \pi \tan^2\alpha (h-H)^2 H$$

$$\frac{dV}{dH} = \pi \tan^2\alpha[(h-H)^2 \times 1 + H \times 2(h-H)(-1)]$$
$$= \pi \tan^2\alpha (h-H)[h - H - 2H]$$
$$= \pi \tan^2\alpha (h-H)[h-3H]$$

For max volume, $\frac{dV}{dH} = 0$

$\Rightarrow h - H = 0$ or $h - 3H = 0$

$\Rightarrow h = H$ (not possible) $\Rightarrow H = h/3$

Now, $\frac{d^2V}{dH^2} = \pi \tan^2\alpha[(h-H)(-3) + (h-3H)(-1)]$

$$= \pi \tan^2\alpha(6H - 4h)$$

< 0, when $H = h/3$

∴ Volume is maximum when $H = h/3$

Now, V= $\pi \tan^2\alpha (h-H)^2 H$

$$= \pi \tan^2\alpha \left(h - \frac{h}{3}\right)^2 . \frac{h}{3} = \frac{4}{27}\pi h^3 \tan^2\alpha$$

HOME WORK

EXAMPLES

42 , 43, 44, 45, 46